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Five Klamkin Quickies (**Olympiad corner n.213**) (**CRUX**, **vol.27**,**n.3**,**2001**)

Quickie 1.

Prove that $a+b+c \geq \sqrt{b^2+c^2-a^2} + \sqrt{c^2+a^2-b^2} + \sqrt{a^2+b^2-c^2}$
where a,b,c are sides of non-obtuse triangle.

Solution by Arkady Alt, **San Jose**,**California**, **USA**.

Let F, s be, respectively, area and semiperimeter of the triangle.

Note that $\left(\sqrt{b^2+c^2-a^2} + \sqrt{c^2+a^2-b^2} + \sqrt{a^2+b^2-c^2}\right)^2 =$
$a^2+b^2+c^2+2\sum\sqrt{(b^2+c^2-a^2)(c^2+a^2-b^2)}$.

Since by AM-QM Inequality
$\sum\sqrt{(b^2+c^2-a^2)(c^2+a^2-b^2)} \leq \sqrt{3\sum(b^2+c^2-a^2)(c^2+a^2-b^2)} =$
$\sqrt{3(2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4)} = \sqrt{3\cdot 16F^2} = 4\sqrt{3}F$ and*

(**1**) $\quad 4\sqrt{3}F \leq ab+bc+ca$

then $\left(\sqrt{b^2+c^2-a^2} + \sqrt{c^2+a^2-b^2} + \sqrt{a^2+b^2-c^2}\right)^2 \leq$
$a^2+b^2+c^2+2(ab+bc+ca) = (a+b+c)^2$.

* Let $x := s-a > 0, y := s-b > 0, z := s-c > 0$ and $p := xy+yz+zx, q := xyz$.
Assuming $s = 1$ we obtain $x+y+z = 1$ and $a = 1-x, b = 1-y, c = 1-z$,
$ab+bc+ca = 1+p, F = \sqrt{q}$.

Then since $3p \leq 1(3(xy+yz+zx) \leq (x+y+z)^2), 3q \leq p^2(3xyz(x+y+z) \leq (xy+yz+zx)^2)$
we obtain $1+p-4\sqrt{3q} \geq 1+p-4\sqrt{3\cdot\dfrac{p^2}{3}} = 1-3p \geq 0.$

Remark.

The following solution, that belong to M. Klamkin, is much more simpler and completely corresponds to the specified type of the problem:

Applying inequality $x+y \leq \sqrt{2(x^2+y^2)}$ we obtain
$2\sum\sqrt{b^2+c^2-a^2} = \sum\left(\sqrt{b^2+c^2-a^2} + \sqrt{c^2+a^2-b^2}\right) \leq$
$\sum\sqrt{2(b^2+c^2-a^2+c^2+a^2-b^2)} = \sum\sqrt{2\cdot 2c^2} = 2(a+b+c).$

Quickie 2.(**Corrected**)

Determine the extreme values of the area of a triangle ABC given the lengths of the two altitudes h_b, h_c.

(In original setting was "two altitudes h_b, h_c and the side $BC = a$", but then $area(ABC) = \dfrac{h_a a}{2}$ is a constant and, therefore, problem have no sense.
Also, if h_b, h_c are given then any fixed $a > 0$ together with h_b, h_c determine triangle and, therefore, its area).

Solution by Arkady Alt, **San Jose**,**California**, **USA**.

Let F and R be, respectively, area and circumradius of $\triangle ABC$.

Since $h_b = \dfrac{c\cdot a}{2R} \Leftrightarrow c = \dfrac{2R\cdot h_b}{a}, h_c = \dfrac{ab}{2R} \Leftrightarrow b = \dfrac{2R\cdot h_c}{a}$ we obtain
$h_a = \dfrac{bc}{2R} = \dfrac{2R\cdot h_b\cdot h_c}{a^2}$ and, therefore, $F = \dfrac{a\cdot h_a}{2} = \dfrac{R\cdot h_b\cdot h_c}{a} =$

$\dfrac{h_b \cdot h_c}{2\sin A}$. Hence, $\min F = \dfrac{h_b \cdot h_c}{2\sin\frac{\pi}{2}} = \dfrac{h_b \cdot h_c}{2}$ and F not bounded from above.

Quickie 3.

Determine the maximum area of a triangle ABC given the perimeter p and the angle A.

Solution by Arkady Alt, San Jose, California, USA.

Let F, s, R and r be, respectively, area, semiperimeter, circumradius and inradius of $\triangle ABC$.

Since $\tan\frac{A}{2} = \frac{r}{s-a}$ then $F = rs = s(s-a)\tan\frac{A}{2}$. Noting that $s = 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ and $2\cos\frac{B}{2}\cos\frac{C}{2} = \cos\frac{B-C}{2} + \cos\frac{B+C}{2} = \cos\frac{B-C}{2} + \sin\frac{A}{2} \le 1 + \sin\frac{A}{2}$ we obtain that $s \le 2R\cos\frac{A}{2}\left(1+\sin\frac{A}{2}\right) \Leftrightarrow 2R \ge \dfrac{s}{\cos\frac{A}{2}\left(1+\sin\frac{A}{2}\right)}$ and, therefore,

$$a = 2R\sin A \ge \frac{s\sin A}{\cos\frac{A}{2}\left(1+\sin\frac{A}{2}\right)} = \frac{2s\sin\frac{A}{2}}{1+\sin\frac{A}{2}}.$$

Hence, $F \le s\left(s - \dfrac{2s\sin\frac{A}{2}}{1+\sin\frac{A}{2}}\right)\tan\frac{A}{2} = \dfrac{s^2\tan\frac{A}{2}\left(1-\sin\frac{A}{2}\right)}{1+\sin\frac{A}{2}} = \dfrac{p^2\tan\frac{A}{2}\left(1-\sin\frac{A}{2}\right)}{4\left(1+\sin\frac{A}{2}\right)}$

Since in the latter inequality equality occurs iff it occurs in $2\cos\frac{B}{2}\cos\frac{C}{2} \le 1+\sin\frac{A}{2}$ (that is iff $B = C$) then $\max F = \dfrac{p^2\tan\frac{A}{2}\left(1-\sin\frac{A}{2}\right)}{4\left(1+\sin\frac{A}{2}\right)} = \dfrac{p^2\tan\frac{A}{2}\left(1-\sin\frac{A}{2}\right)^2}{4\cos^2\frac{A}{2}}$.

Solution 2.

Since $a^2 = b^2 + c^2 - 2bc\cos A, b^2 + c^2 \ge \dfrac{(b+c)^2}{2}$ and $bc \le \dfrac{(b+c)^2}{4}$ then

$a^2 \ge \dfrac{(b+c)^2}{2} - \dfrac{(b+c)^2}{2}\cos A = (b+c)^2\sin^2\frac{A}{2} \Leftrightarrow a \ge (b+c)\sin\frac{A}{2} \Leftrightarrow$

$p - (b+c) \ge (b+c)\sin\frac{A}{2} \Leftrightarrow b + c \le \dfrac{p}{1+\sin\frac{A}{2}}$.

Hence, $F = \dfrac{bc}{2}\sin A \le \dfrac{(b+c)^2}{8}\sin A \le \dfrac{p^2\sin A}{8\left(1+\sin\frac{A}{2}\right)^2} = \dfrac{p^2\sin\frac{A}{2}\cos\frac{A}{2}}{4\left(1+\sin\frac{A}{2}\right)^2} =$

$\dfrac{p^2\tan\frac{A}{2}\cos^2\frac{A}{2}}{4\left(1+\sin\frac{A}{2}\right)^2} = \dfrac{p^2\tan\frac{A}{2}\left(1-\sin^2\frac{A}{2}\right)}{4\left(1+\sin\frac{A}{2}\right)^2} = \dfrac{p^2\tan\frac{A}{2}\left(1-\sin\frac{A}{2}\right)}{4\left(1+\sin\frac{A}{2}\right)}$ and since

in inequality $F \le \dfrac{p^2\tan\frac{A}{2}\left(1-\sin\frac{A}{2}\right)}{4\left(1+\sin\frac{A}{2}\right)}$ equality occurs iff $b = c$ then

$$\max F = \frac{p^2\tan\frac{A}{2}\left(1-\sin\frac{A}{2}\right)}{4\left(1+\sin\frac{A}{2}\right)}$$

Quickie 4

Determine the minimum value of

$$\sum\left(\frac{a_2+a_3+a_4+a_5}{a_1}\right)^{1/2}$$

where the sum is cyclic over the positive numbers a_1, a_2, a_3, a_4, a_5.

Solution by Arkady Alt, **San Jose**,**California**, **USA**.

Twice using AM-GM Ineequality we obtain

$$\sum\left(\frac{a_2+a_3+a_4+a_5}{a_1}\right)^{1/2} \geq \sum\left(\frac{4(a_2a_3a_4a_5)^{1/4}}{a_1}\right)^{1/2} =$$

$$2\sum\left(\frac{a_2a_3a_4a_5}{a_1^4}\right)^{1/8} \geq 2\cdot 5\left(\prod\left(\frac{a_2a_3a_4a_5}{a_1^4}\right)^{1/8}\right)^{1/5} =$$

$10\left(\prod \frac{a_2a_3a_4a_5}{a_1^4}\right)^{1/40} = 10$ because $\prod a_2a_3a_4a_5 = a_1^4a_2^4a_3^4a_4^4a_5^4$.